Final Exam Practice Problems

1. For the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 6 & 1 \\ 7 & 0 \\ 8 & -2 \end{bmatrix}, C = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$, calculate:

(i) $6A + 2B^t$.
(ii) $\det(2B \cdot 3A)$.
(iii) ABC^{-1}.

Solution. (i) $6A + 2B^t = \begin{bmatrix} 6 & 12 & 18 \\ 30 & -6 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 14 & 16 \\ 2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 18 & 26 & 34 \\ 32 & -6 & -4 \end{bmatrix}$.

(ii) $\det(2B \cdot 3A) = \det(\begin{bmatrix} 12 & 2 \\ 14 & 0 \\ 16 & -4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 & 9 \\ 15 & -3 & 0 \end{bmatrix} = \det \begin{bmatrix} 66 & 66 & 108 \\ 42 & 84 & 126 \\ -12 & 108 & 144 \end{bmatrix} = 0.$

(iii) $AB = \begin{bmatrix} 44 & -5 \\ 23 & 5 \end{bmatrix}, C^{-1} = \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix}, ABC^{-1} = \begin{bmatrix} -113 & 167 \\ -21 & 34 \end{bmatrix}$.

2. Write the solution set to the system of linear equations having the following augmented matrices in reduced row echelon form:

$$A = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 8 \end{array}\right].$$

Solution. For A : $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} s \\ 5-3t \\ 7-6t \\ t \end{bmatrix}$, such that $s, t \in \mathbb{R}$. For B : $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2-t \\ t \end{bmatrix}$, for $t \in \mathbb{R}$. For C: No solution.

3. Suppose that U is the 3×3 coefficient matrix of the system of equations

$$\begin{aligned} x + 2y + 3z &= 2 \\ y + 4z &= -1 \\ 5x + 6z &= -5. \end{aligned}$$

(i) Calculate the determinant of U in two ways: By expanding along the third row and expanding along the second column.
(ii) Convert the system of equations into a single matrix equation.
(iii) Solve the matrix equation by finding U^{-1}.
(iv) Write the solution to the original systems of equation.

(i) Expanding along the 3rd row: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 0 & 6 \end{vmatrix} = 5 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} + 6 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 5 \cdot 5 - 0 + 6 \cdot 1 = 31.$

Expanding along the second column: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 0 & 6 \end{vmatrix} = -2 \cdot \begin{vmatrix} 0 & 4 \\ 5 & 6 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 40 + (-9) - 0 = 31.$

(ii) For $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$, we have $U \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$.

(iii) Applying elementary row operations to $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 0 & 6 & 0 & 0 & 1 \end{array}\right]$ yields $U^{-1} = \frac{1}{31} \cdot \begin{bmatrix} 6 & -12 & 5 \\ 20 & -9 & -4 \\ -5 & 10 & 1 \end{bmatrix}$.

4. Determine whether or not the following matrices are diagonalizable. If not, explain why. If so, in each case find the diagonalizing matrix P. In these latter cases, check your answer, i.e., if P diagonalizes say A, then verify that $P^{-1}AP$ is a diagonal matrix.

$$A = \begin{bmatrix} -4 & 2 \\ 0 & 6 \end{bmatrix}, B = \begin{bmatrix} -1 & -2 & 2 \\ 4 & 3 & -4 \\ 0 & -2 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -13 & 0 & 16 \\ 0 & 0 & 3 & 0 \\ 0 & -12 & 0 & 15 \end{bmatrix}.$$

In finding the eigenvalues of each matrix, it might be useful to use the following version of the Rational Root Test: If $p(x) = x^n + a_1x^{n-1} + \cdots + a_n$ is a polynomial with integer coefficients, then the only possible roots in the rational numbers are $\pm d$ where d divides a_n.

Solution. (i) For A: $c_A(x) = (x+4)(x-6)$, so A has distinct eigenvalues $\lambda = -4, 6$ and is therefore diagonalizable. For $\lambda : A - (-4)I_2 = \begin{bmatrix} -4 & 2 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$, which reduces to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the corresponding eigenvector. For $\lambda = 6$, we consider $\begin{bmatrix} -4 & 2 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} -10 & 2 \\ 0 & 0 \end{bmatrix}$ which reduces to $\begin{bmatrix} -5 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is the corresponding eigenvector. Therefore $P = \begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix}$ is the diagonalizing matrix, with $P^{-1} = \frac{1}{5} \cdot \begin{bmatrix} 5 & -1 \\ 0 & 1 \end{bmatrix}$.

(ii) For $B : c_B(x) = x^3 - 3x^2 - x + 3 = (x-1)(x+1)(x-3)$. Thus, B has distinct eigenvalues 1, -1, 3 and is therefore diagonalizable. For $\lambda = 1 \ : B - (1 \cdot I_3) = \begin{bmatrix} -2 & -2 & 2 \\ 4 & 2 & -4 \\ 0 & -2 & 0 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the corresponding eigenvector. For $\lambda = -1 : B - (-1 \cdot I_3) = \begin{bmatrix} 0 & -2 & 2 \\ 4 & 4 & -4 \\ 0 & -2 & 2 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so that $v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is the corresponding eigenvector. For $\lambda = 3 : B - 3 \cdot I_3 = \begin{bmatrix} -4 & -2 & 2 \\ 4 & 0 & -4 \\ 0 & -2 & -2 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so that $v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the corresponding eigenvector. Thus, the diagonalizing matrix is $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, with $P^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}$.

(iii) For C: $c_C(x) = (X = 2)^2(x-1)$, so the eignevalues are -2 (with multiplicity 2) and 1. For $\lambda = 2 :$ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are a basis for E_2, the eigenspace of 2. For $\lambda = 1 :$ $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is a basis for E_1. Thus, C is diagonalizable, with diagonalizing matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$, with $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(iv) For D: $c_D(x) = (x-2)^2(x-1)$. $\lambda = 2$ has multiplicity 2, but an easy calculation shows that the dimension of E_1 is 1, thus, D is not diagonalizable.

(v) For E: $c_E(x) = x^4 - 8x^3 + 18x^2 - 27 = (x-3)^3(x+1)$. For $\lambda = 3$, which has multiplicity 3: The usual calculation would show that E_3 has dimension three with a basis $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$. Note there are many possible bases, but any basis you find should have three elements. An eigenvector for -1 is $\begin{bmatrix}0\\4\\0\\4\end{bmatrix}$. Thus E is diagonalizable, with diagonalizing matrix $P = \begin{bmatrix}1&0&0&0\\0&1&0&4\\0&0&1&0\\0&1&0&3\end{bmatrix}$, with $P^{-1} = \frac{1}{26}\begin{bmatrix}1&0&0&0\\0&3&0&4\\0&0&26&0\\0&-1&0&10\end{bmatrix}$.

5. For the matrices in problem 4 that are diagonalizable, say, for example, A, write a formula for A^n and e^A.

Solution. I did not realize that so many of the matrices in the previous problem would be diagonalizable. In any case, for any diagonalizable matrix M with diagonalizing matrix P, if $P^{-1}MP = D = \begin{bmatrix}\lambda_1&0&0&\cdots&0\\0&\lambda_2&\cdots&0&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\cdots&0&\lambda_n\end{bmatrix}$, then $M^k = PD^kP^{-1}$ and $e^M = Pe^DP^{-1}$, where $D^k = \begin{bmatrix}\lambda_1^k&0&0&\cdots&0\\0&\lambda_2^k&\cdots&0&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\cdots&0&\lambda_n^k\end{bmatrix}$, for all $k \geq 1$ and $e^D = \begin{bmatrix}e^{\lambda_1}&0&0&\cdots&0\\0&e^{\lambda_2}&\cdots&0&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\cdots&0&e^{\lambda_n}\end{bmatrix}$.

For A: $A^n = \begin{bmatrix}(-4)^n & \frac{(-1)^{n+1}4^n+6^n}{5}\\0&6^n\end{bmatrix}$ and $e^A = \begin{bmatrix}e^{-4} & \frac{-e^4}{5}+\frac{e^6}{5}\\0&e^6\end{bmatrix}$.

For B: $B^n = \begin{bmatrix}-3^n+2 & -3^n+1 & 3^n-1\\(-1)^{n+1}+3^n & 3^n & (-1)^n-3^n\\(-1)^n-3^n+2 & -3^n+1 & (-1)^n+3^n-1\end{bmatrix}$ and $e^B = \begin{bmatrix}2e-e^3 & e-e^3 & -e+e^3\\-e^{-1}+e^3 & e^3 & e^{-1}-e^3\\2e-e^{-1}-e^3 & e-e^3 & -e+e^{-1}+e^3\end{bmatrix}$.

6. For the system of first order linear differential equations:

$$x_1'(t) = 2x_1(t)$$
$$x_2'(t) = x_1(t) + x_2(t)$$
$$x_3'(t) = -x_1(t) + x_3(t)$$

with initial conditions $x_1(0) = -1, x_2(0) = -2, x_3(0) = 2$.

(i) Convert the system of equations into a single matrix differential equation with initial condition, clearly indicating the terms the matrix equation.
(ii) Solve the equation in (ii) using the exponential function.
(iii) Write the solutions to the original system.

Solution. (i) Set $A := \begin{bmatrix}2&0&0\\1&1&0\\-1&0&1\end{bmatrix}$, then the matrix equation is $\mathbf{X}'(t) = A\cdot\mathbf{X}(t)$, where $\mathbf{X}'(t) = \begin{bmatrix}x_1'(t)\\x_2'(t)\\x_3'(t)\end{bmatrix}$ and $\mathbf{X}(t) = \begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}$. The initial conditions are $\mathbf{X}(0) = \begin{bmatrix}-1\\-2\\2\end{bmatrix}$.

In matrix form, the solution to the system is given by $\mathbf{X}(t) = e^{At} \cdot \mathbf{X}(0)$. For e^{At}, one calculates the following: $c_A(x) = (x-1)^2(x-2)$. A basis for E_1 can be taken to be: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and a basis for E_2 is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Thus, the diagonalizing matrix and its inverse are: $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. When then have:

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 0 & 0 \\ -e^t + e^{2t} & e^t & 0 \\ e^t - e^{2t} & 0 & e^2 \end{bmatrix}. \end{aligned}$$

Thus,

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & 0 & 0 \\ -e^t + e^{2t} & e^t & 0 \\ e^t - e^{2t} & 0 & e^2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -e^{2t} \\ -e^t - e^{2t} \\ e^t + e^{2t} \end{bmatrix}$$

and therefore, $x_1(t) = -e^{2t}, x_2(t) = -e^t - e^{2t}$, and $x_3(t) = e^t + e^{2t}$.

7. Given the linear recurrence relation $a_{k+2} = a_{k+1} + 2a_k$ and the initial conditions $a_0 = 2, a_1 = 7$,
   (i) Write out the first five terms of the sequence.
   (ii) Find a formula for a_k, for all k, that is not recursive.

Solution. (i) $a_0 = 2, a_1 = 7, a_2 = 11, a_3 = 25, a_4 = 47$.

(ii) Take $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. We have $\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = A^k \cdot \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = A^k \cdot \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, for all $k \geq 1$. Recall that $A^k = PD^kP^{-1}$ where, P is the diagonalizing matrix for A and D is the diagonal form of A. For this: $c_A(x) = (x-2)(x+1)$,$P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $P^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Then:

$$A^k = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 1 * 1 & \\ 2 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^k + 2(-1)^k & 2^k - (-1)^k \\ 2^{k+1} - 2(-1)^k & 2^{k+1} + (-1)^k \end{bmatrix}.$$

For a_k, we just need the first entry of the vector $A^k \cdot \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, which gives $a_k = \frac{1}{3} \cdot (2^{k+1} + 4 \cdot (-1)^k + 7 \cdot 2^k - 7 \cdot (-1)^k)$

8. Let U be the subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ -2 \\ 8 \\ 6 \end{bmatrix}, v_4 = \begin{bmatrix} 8 \\ -1 \\ 9 \\ 8 \end{bmatrix}$.
   (i) Find a basis for U.
   (ii) Determine if the vectors $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \end{bmatrix}$ belong to U.

Solution. To find a basis for U we have to find redundant vectors and remove them. For this we have to find non-trivial linear combinations of v_1, v_2, v_3, v_4 and for this, no-trivial solutions to the homogeneous system with coefficient matrix $\begin{bmatrix} 1 & 0 & 6 & 8 \\ 0 & 1 & -2 & -1 \\ 1 & -1 & 8 & 9 \\ 1 & 0 & 6 & 8 \end{bmatrix}$. Elementary row operations leads to the reduced row echelon form $\begin{bmatrix} 1 & 0 & 6 & 8 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. From this, the solutions to the system are $\begin{bmatrix} -6s-8t \\ 2s+t \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} -6 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -8 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Thus, $-6v_1 + v_2 + v_3 = 0$ and $-8v_1 + v_2 + v_3 = 0$, from which it follows that $v_3 = 6v_1 - v_2$ and $v_4 = 8v_1 - v_2$, so that v_2, v_4 are redundant. Thus, v_1 and v_2 form a basis for U. (Note v_2 is not a multiple of v_1, so v_1 and v_2 are linearly independent.)

9. Find an orthonormal basis for each of the eigenspaces of the diagonalizable matrices in problem 4.

Solution. It turns out this is not a very interesting problem, since many of the bases found are already orthogonal, or the eigenspaces are just one dimensional.

For A, each eigenspace is one dimensional, so orthogonality within a given eigenspace is not an issue. Same comment for B. For C, the given eigenvectors for E_2 are already orthogonal, and E_1 is one dimensional. D is not diagonalizable. For E, the given vectors for E_3 are mutually orthogonal and E_{-1} is one dimensional. In all cases, once one has an orthogonal basis, one creates an orthonormal basis by dividing each vector by its length.

10. Find the best approximation to a solution of the system

$$\begin{aligned} 6x + 4y &= 14 \\ 8x - 2y &= 4 \\ -2x + 4y &= 4. \end{aligned}$$

Solution. First note that $x = 1$ and $y = 2$ is the unique solution to the first two equations yet they are not a solution to the third equation. Therefore the given system does not have a solution. The two methods to finding a best approximate solutions are are follows:

(i) Set $w_1 = \begin{bmatrix} 6 \\ 8 \\ -2 \end{bmatrix}$, $w_2 = \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 4 \\ 4 \end{bmatrix}$, and $U = \text{span}\{w_1, w_2\}$. A best approximate solution to the given system is obtained by solving the system $A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{p}_U B$, where A is the matrix whose columns are w_1, w_2 and $\mathbf{p}_U B$ is the orthogonal projection of B onto U. By definition,

$$\mathbf{p}_U B = \frac{w_1 \cdot B}{w_1 \cdot w_1} w_1 + \frac{w_2 \cdot B}{w_2 \cdot w_2} w_2 = \frac{27}{26} \begin{bmatrix} 6 \\ 8 \\ -2 \end{bmatrix} + \frac{23}{29} \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}.$$

Alternately, one may solve the system $A^t \cdot A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^t B$. In this case, $A^t A = \begin{bmatrix} 104 & 0 \\ 0 & 36 \end{bmatrix}$ while $A^t B = \begin{bmatrix} 108 \\ 64 \end{bmatrix}$, so that $\begin{bmatrix} 104 & 0 \\ 0 & 36 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 108 \\ 64 \end{bmatrix}$ from which it follows that $x = \frac{108}{104}$ and $y = \frac{64}{36}$. Note that since AA^t is invertible, there is a **unique** best approximate solution to the original system In general, there may be multiple equally good best approximations.

11. Given the data points (2,6), (-1,4), (-2,3), find:

(i) The line best fitting the data.
(ii) The quadratic polynomial best fitting the data.

Solution. (i) For the line of best fit, we start with $y = mx + b$ and use the data points to find the best approximation to a solution of the system

$$\begin{aligned} 6 &= b + m2 \\ 4 &= b + m(-1) \\ 3 &= b + m(-2) \end{aligned}$$

In matrix terms, we have $A \cdot \begin{bmatrix} b \\ m \end{bmatrix} = v$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -2 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$. Thus, we have to solve $A^t A \cdot \begin{bmatrix} b \\ m \end{bmatrix} = A^t \cdot \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$. Thus latter equation is: $\begin{bmatrix} 3 & -1 \\ -1 & 9 \end{bmatrix} \cdot \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}$. Multiplying bythe inverse of A we get

$$\begin{bmatrix} b \\ m \end{bmatrix} = \frac{1}{26} \cdot \begin{bmatrix} 9 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 2 \end{bmatrix} = \frac{1}{26} \cdot \begin{bmatrix} 119 \\ 19 \end{bmatrix}.$$

Therefore the line best fitting the given data is: $y = \frac{19}{26}x + \frac{119}{26}$.

(ii) For the best fit quadratic polynomial $y = a_0 + a_1 x + a_2 x^2$, substituting the data points gives:

$$\begin{aligned} 6 &= a_0 + a_1 2 + a_2 2^2 \\ 4 &= a_0 + a_1(-1) + a_2(-1)^2 \\ 3 &= a_0 + a_1(-2) + a_2(-2)^2 \end{aligned}$$

Writing $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$ we want a best approximate solution to the system $A \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$ $(\star)$. Multiplying by A^t, we get $\begin{bmatrix} 3 & -1 & 9 \\ -1 & 9 & -1 \\ 9 & -1 & 33 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 40 \end{bmatrix}$. Multiplying both sides of this last equation by $(A^t A)^{-1}$, we get $a_0 = \frac{29}{6}, a_1 = \frac{3}{4}, a_2 = -\frac{1}{12}$. Thus, the quadratic best fitting the data is $y = \frac{29}{6} + \frac{3}{4}x - \frac{1}{12}x^2$. **Note:** In this example, the matrix A is invertible so that the original system $(\star)$ has an exact solution. Thus the quadratic polynomial we found passes through each of the three given data points.